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On Quadruple Theta-Functions.

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PART II.

In the following I employ the notation used by Schottky in his "*Abriss einer Theorie der Abelschen Functionen von drei Variabeln.*" On page 18, Schottky gives the fundamental theorem bearing upon his particular notation; it is as follows: "*Es ist möglich, ein System primitiver Indices*

$$1, 2, 3 \ldots 2\rho + 1$$

und einen ausgezeichneten ε *so zu wählen, dass* εa *ein grader Index ist, wenn die Anzahl der primitiven Indices, aus denen* a *zusammengesetzt ist* $\equiv \rho$ *oder* $\rho + 1$ *mod.* 4 *ist, dagegen ein ungrader, wenn diese Anzahl* $\equiv \rho + 2$ *oder* $\rho - 1$ *mod.* 4 *ist.*"

For $\rho = 4$ the "primitive indices" are nine in number, and they may in general be denoted by the letters

$$k, l, m, n, p, q, r, s, t.$$

All of the characteristics of the quadruple theta-functions, with the exception of (0), may be represented by certain combinations of these letters, viz. by taking them one at a time, two at a time, three at a time and four at a time. We have thus:

	Number of cases.
The index (0)	1
The primitive indices taking one at a time	9
" " " two "	36
" " " three "	84
" " " four "	126
	256

The even functions, 136 in number, are given by the first, second and fifth of these cases, and the odd functions, 120 in number, by the third and fourth cases. That is, the even functions will have the suffixes o, k and $klmn$, and the odd functions will have the suffixes kl and klm. In the numbers of the *Annales de* *l'École Normale* for June, July and August, 1883, M. Brunel has investigated the relations similar to the Göpel and Kummer relations for the double theta-functions which exist in the case of the triple theta-functions. I propose in what follows to employ Brunel's method in working out the corresponding relations connecting the quadruple functions. Brunel starts out from certain relations given by Schottky in the *Nachtrag* to the above mentioned book "*Ueber die hyperelliptischen Functionen dreier Variabeln,*" and uses a method which is fundamentally the same as that employed by Brioschi in his paper already referred to in Part I of this article,* but the manner in which he develops it is simpler than would be possible had he employed, without alteration, the method indicated by Brioschi.

I shall use almost without change the notation employed by Brunel, only altering it when the greater complexity of the present case makes it desirable. Following Schottky, write first:

$$\frac{L_k^4}{L_0^4}=\frac{-1}{(a_l-a_k)(a_m-a_k)(a_n-a_k)(a_p-a_k)(a_q-a_k)(a_r-a_k)(a_s-a_k)(a_t-a_k)}$$

$$\frac{L_{kl}^4}{L_0^4}=\frac{-1}{\left\{\begin{array}{l}(a_m-a_k)(a_n-a_k)(a_p-a_k)(a_q-a_k)(a_r-a_k)(a_s-a_k)(a_t-a_k)\\ \times(a_m-a_l)(a_n-a_l)(a_p-a_l)(a_q-a_l)(a_r-a_l)(a_s-a_l)(a_t-a_l)\end{array}\right.}$$

$$\frac{L_{klm}^4}{L_0^4}=\frac{-1}{\left\{\begin{array}{l}(a_n-a_k)(a_p-a_k)(a_q-a_k)(a_r-a_k)(a_s-a_k)(a_t-a_k)\\ \times(a_n-a_l)(a_p-a_l)(a_q-a_l)(a_r-a_l)(a_s-a_l)(a_t-a_l)\\ \times(a_n-a_m)(a_p-a_m)(a_q-a_m)(a_r-a_m)(a_s-a_m)(a_t-a_m)\end{array}\right.}$$

$$\frac{L_{klmn}^4}{L_0^4}=\frac{-1}{\left\{\begin{array}{l}(a_p-a_k)(a_q-a_k)(a_r-a_k)(a_s-a_k)(a_t-a_k)\\ \times(a_p-a_l)(a_q-a_l)(a_r-a_l)(a_s-a_l)(a_t-a_l)\\ \times(a_p-a_m)(a_q-a_m)(a_r-a_m)(a_s-a_m)(a_t-a_m)\\ \times(a_p-a_n)(a_q-a_n)(a_r-a_n)(a_s-a_n)(a_t-a_n)\end{array}\right.}$$

Now consider the functions P defined by the equations

1. $$\frac{L_0}{L_0}P_0=\frac{\theta_0}{\theta_0},\quad \frac{L_k}{L_0}P_k=\frac{\theta_k}{\theta_0},$$
$$\frac{L_{kl}}{L_0}P_{kl}=\frac{\theta_{kl}}{\theta_0},\quad \frac{L_{klm}}{L_0}P_{klm}=\frac{\theta_{klm}}{\theta_0},$$
$$\frac{L_{klmn}}{L_0}P_{klmn}=\frac{\theta_{klmn}}{\theta_0},$$

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* Brioschi.—*La relazione di Göpel per funzioni iperellittiche d'ordine qualunque.* Annali di Matematica, Serie II^a, Tomo X°.

then, following Weierstrass and Schottky, and writing

$$2.\quad \begin{aligned} R(x) &= (a_k - x)(a_l - x)(a_m - x)(a_n - x)(a_p - x)(a_q - x)(a_r - x)(a_s - x)(a_t - x) \\ \phi(x) &= (x - x_1)(x - x_2)(x - x_3)(x - x_4) \end{aligned}$$

we have

$$P_0 = 1 \;.\; 1 \text{ function } P_0$$

$$P_k = \sqrt{(a_k - x_1)(a_k - x_2)(a_k - x_3)(a_k - x_4)} \;.\;.\;.\;.\;.\;.\; 9 \text{ functions } P_k$$

$$3.\quad P_{kl} = P_k P_l \sum_{i=1}^{i=4} \frac{\sqrt{R(x_i)}}{(a_k - x_i)(a_l - x_i)\,\varphi'(x_i)} \;.\;.\;.\;.\;.\;.\;.\;.\;.\; 36 \text{ functions } P_{kl}$$

$$P_{klm} = P_k P_l P_m \sum_{i=1}^{i=4} \frac{\sqrt{R(x_i)}}{(a_k - x_i)(a_l - x_i)(a_m - x_i)\,\varphi'(x_i)} \;.\;.\;.\;.\;.\; 84 \text{ functions } P_{klm}$$

$$P_{klmn} = P_k P_l P_m P_n \sum_{i=1}^{i=4} \frac{\sqrt{R(x_i)}}{(a_k - x_i)(a_l - x_i)(a_m - x_i)(a_n - x_i)\,\varphi'(x_i)} \;.\;.\;. 126 \text{ functions } P_{klmn}$$

making in all 256 P-functions replacing the 256 Θ-functions. In these equations the letters k, l, m and n are all different from each other. We have now to determine the linear relations existing between the squares of these P-functions and those existing between their products taken two and two. Write

$$4.\qquad \Sigma x_i = \alpha, \quad \Sigma x_i x_j = \beta, \quad \Sigma x_i x_j x_k = \gamma, \quad x_1 x_2 x_3 x_4 = \delta.$$

The summations to be taken from 1 to 4 and i, j, k all having different values. Further write

$$(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4) = -\theta,$$

that is

$$5.\qquad \begin{vmatrix} x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -\theta;$$

and in general write

$$\begin{vmatrix} y_1^{n-1} & y_2^{n-1} & \cdots & y_n^{n-1} \\ y_1^{n-2} & y_2^{n-2} & \cdots & y_n^{n-2} \\ \cdot & \cdot & \cdots & \cdot \\ y_1 & y_2 & \cdots & y_n \\ 1 & 1 & \cdots & 1 \end{vmatrix} \equiv |y_1 y_2 \ldots y_n|,$$

that is

$$6.\qquad |x_1 x_2 x_3 x_4| = -\theta.$$

The P-functions can now be written in the following manner,

$$P_0 = 1, \quad P_k = \sqrt{|a_k x_1||a_k x_2||a_k x_3||a_k x_4|}$$

$$P_{kl} = \frac{1}{|x_1 x_2 x_3 x_4|}\left\{ |x_2 x_3 x_4| \sqrt{R(x_1)\frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4|}{|a_k x_1||a_l x_1|}} \right.$$
$$\left. - |x_3 x_4 x_1| \sqrt{\ } \qquad + |x_4 x_1 x_2| \sqrt{\ } \qquad - |x_1 x_2 x_3| \sqrt{\ } \right\}$$

7.
$$P_{klm} = \frac{1}{|x_1 x_2 x_3 x_4|}\left\{ |x_2 x_3 x_4| \sqrt{R(x_1)\frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4||a_m x_2||a_m x_3||a_m x_4|}{|a_k x_1||a_l x_1||a_m x_1|}} \right.$$
$$\left. - |x_3 x_4 x_1| \sqrt{\ } \qquad + |x_4 x_1 x_2| \sqrt{\ } \qquad - |x_1 x_2 x_3| \sqrt{\ } \right\}$$

$$P_{klmn} =$$
$$\frac{1}{|x_1 x_2 x_3 x_4|}\left\{ |x_2 x_3 x_4| \sqrt{R(x_1)\frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4||a_m x_2||a_m x_3||a_m x_4||a_n x_2||a_n x_3||a_n x_4|}{|a_k x_1||a_l x_1||a_m x_1||a_n x_1|}} \right.$$
$$\left. - |x_3 x_4 x_1| \sqrt{\ } \qquad + |x_4 x_1 x_2| \sqrt{\ } \qquad - |x_1 x_2 x_3| \sqrt{\ } \right\}$$

It is of course perfectly obvious how to fill up the empty radical signs. The above forms for the P-functions are retained for the same reason that Brunel gives in the case of the triple theta-functions, that is, although the denominators under the radical signs are actually factors of $R(x_1)$, $R(x_2)$, $R(x_3)$ and $R(x_4)$, it is more convenient in the following transformations to retain the fractional form.

LINEAR RELATIONS BETWEEN THE SQUARES OF THE P-FUNCTIONS.

Take first the case of the functions with a single index, *i. e.* P_k, P_l, etc. We have

8.
$$P_k^2 = |a_k x_1||a_k x_2||a_k x_3||a_k x_4|;$$

expanding this and using the notation given above, we have

9.
$$P_k^2 = a_k^4 - \alpha a_k^3 + \beta a_k^2 - \gamma a_k + \delta.$$

As this is linear in the quantities α, β, γ, δ, and as k is any one of the primitive indices, we can, by assuming any five such relations, eliminate α, β, γ and δ; the result of the elimination is obviously

10.
$$\begin{vmatrix} P_k^2 - a_k^4, & P_l^2 - a_l^4, & P_m^2 - a_m^4, & P_n^2 - a_n^4, & P_p^2 - a_p^4 \\ a_k^3 & a_l^3 & a_m^3 & a_n^3 & a_p^3 \\ a_k^2 & a_l^2 & a_m^2 & a_n^2 & a_p^2 \\ a_k & a_l & a_m & a_n & a_p \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0,$$

or expanding this we have

11,
$$P_k^2\,|a_l a_m a_n a_p| + P_l^2\,|a_m a_n a_p a_k| + P_m^2\,|a_n a_p a_k a_l|$$
$$+ P_n^2\,|a_p a_k a_l a_m| + P_p^2\,|a_k a_l a_m a_n| = |a_k a_l a_m a_n a_p|\,P_0^2$$

Since $P_0 = 1$, this factor may be introduced solely for the sake of symmetry. If instead of eliminating α, β, γ and δ between five relations of the form (9) we eliminate α, β, γ, δ and 1 between six such relations, we have obviously

$$12. \qquad \begin{vmatrix} P_k^2 & P_l^2 & P_m^2 & P_n^2 & P_p^2 & P_q^2 \\ a_k^4 & a_l^4 & a_m^4 & a_n^4 & a_p^4 & a_q^4 \\ a_k^3 & a_l^3 & a_m^3 & a_n^3 & a_p^3 & a_q^3 \\ a_k^2 & a_l^2 & a_m^2 & a_n^2 & a_p^2 & a_q^2 \\ a_k & a_l & a_m & a_n & a_p & a_q \\ 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0,$$

or expanding

$$13. \qquad P_k^2 \,|a_l a_m a_n a_p a_q| - P_l^2 \,|a_m a_n a_p a_q a_k| + P_m^2 \,|a_n a_p a_q a_k a_l|$$
$$- P_n^2 \,|a_p a_q a_k a_l a_m| + P_p^2 \,|a_q a_k a_l a_m a_n| - P_q^2 \,|a_k a_l a_m a_n a_p| = 0.$$

We have thus found the linear relations existing between the squares of the P-functions possessing a single suffix, or index, *i. e.* between the functions whose indices are

$$0 \;\; k \;\; l \;\; m \;\; n \;\; p \;\; q \;\; r \;\; s \;\; t,$$

and it is seen that these functions form a group of ten such that any five being given the square of any one of the remaining five can be expressed as a linear function of the squares of the chosen five. Following Brunel I shall call this the group 0.

Consider next the case of the P-functions with two suffixes: for the square of any one of them, say P_{kl}, we have

$$14. \qquad P_{kl}^2 = \frac{1}{|x_1x_2x_3x_4|^2}\Big\{ |x_2x_3x_4|^2 . R(x_1) \frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4|}{|a_kx_1||a_lx_1|} + \ldots$$
$$- 2\,|x_2x_3x_4||x_3x_4x_1|\sqrt{R(x_1)R(x_2)}\,|a_kx_3||a_kx_4||a_lx_3||a_lx_4|$$
$$+ 2\,|x_2x_3x_4||x_4x_1x_2|\sqrt{R(x_1)R(x_3)}\,|a_kx_4||a_kx_2||a_lx_4||a_lx_2|$$
$$- 2\,|x_2x_3x_4||x_1x_2x_3|\sqrt{R(x_1)R(x_4)}\,|a_kx_2||a_kx_3||a_lx_2||a_lx_3|$$
$$- 2\,|x_3x_4x_1||x_4x_1x_2|\sqrt{R(x_2)R(x_3)}\,|a_kx_1||a_kx_4||a_lx_1||a_lx_4|$$
$$+ 2\,|x_3x_4x_1||x_1x_2x_3|\sqrt{R(x_2)R(x_4)}\,|a_kx_1||a_kx_3||a_lx_1||a_lx_3|$$
$$- 2\,|x_4x_1x_2||x_1x_2x_3|\sqrt{R(x_3)R(x_4)}\,|a_kx_1||a_kx_2||a_lx_1||a_lx_2| \Big\}.$$

It is possible to find a linear relation between four of these P-functions with two suffixes which is entirely rational, that is, a relation which shall not contain any of the quantities $\sqrt{R(x_i)R(x_j)}$. Take four of the functions P_{ki} which have the first suffix k in common, say P_{kl}, P_{km}, P_{kn}, P_{kp}, then in order that the radicals $\sqrt{R(x_i)R(x_j)}$ may disappear we must find a series of multipliers A, B, C, D, satisfying the equation

$$15. \qquad A\,|a_lx_3||a_lx_4| - B\,|a_mx_3||a_mx_4| + C\,|a_nx_3||a_nx_4| - D\,|a_px_3||a_px_4| = 0.$$

Giving A, B, C and D the following values,

$$16. \qquad \begin{aligned} A &= |a_m a_n a_p|, \quad & B &= |a_n a_p a_l| \\ C &= |a_p a_l a_m|, \quad & D &= |a_l a_m a_n| \end{aligned}$$

it is easy to see that equation 15 is satisfied. Assuming then four equations of the same form as 14, we have

$$17. \qquad \begin{aligned} & |a_m a_n a_p| P^2_{kl} - |a_n a_p a_l| P^2_{km} + |a_p a_k a_m| P^2_{kn} - |a_l a_m a_n| P^2_{kp} = \\ & \frac{1}{|x_1 x_2 x_3 x_4|^2}\Big[|a_m a_n a_p| \left\{ |x_2 x_3 x_4|^2 R(x_1) \frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4|}{|a_k x_1||a_l x_1|} + \ldots \right\} \\ & \qquad - |a_n a_p a_l| \left\{ |x_2 x_3 x_4|^2 R(x_1) \frac{|a_k x_2||a_k x_3||a_k x_4||a_m x_2||a_m x_3||a_m x_4|}{|a_k x_1||a_m x_1|} + \ldots \right\} \\ & \qquad + |a_p a_l a_m| \left\{ |x_2 x_3 x_4|^2 R(x_1) \frac{|a_k x_2||a_k x_3||a_k x_4||a_n x_2||a_n x_3||a_n x_4|}{|a_k x_1||a_n x_1|} + \ldots \right\} \\ & \qquad - |a_l a_m a_n| \left\{ |x_2 x_3 x_4|^2 R(x_1) \frac{|a_k x_2||a_k x_3||a_k x_4||a_p x_2||a_p x_3||a_p x_4|}{|a_k x_1||a_p x_1|} + \ldots \right\} \Big]. \end{aligned}$$

Introducing the values of $R(x_1)$, $R(x_2)$, etc., it is not difficult to see that the first line of this equation may be thrown into the following form,

$$\begin{aligned} \frac{1}{|x_1 x_2 x_3 x_4|^2} \{ & |a_m a_n a_p||a_m x_1||a_n x_1||a_p x_1| \,[\, |x_2 x_3 x_4||a_q x_1||a_r x_1||a_s x_1||a_t x_1||a_k x_2||a_k x_3||a_k x_4| \,]\, \mathfrak{A} \\ & - |a_n a_p a_l||a_n x_1||a_p x_1||a_l x_1| \,[\quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad]\, \mathfrak{B} \\ & + |a_p a_l a_m||a_p x_1||a_l x_1||a_m x_1| \,[\quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad]\, \mathfrak{C} \\ & - |a_l a_m a_n||a_l x_1||a_m x_1||a_n x_1| \,[\quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad]\, \mathfrak{D} \end{aligned}$$

where
$$\begin{aligned} \mathfrak{A} = {} & |x_2 x_3 x_4||a_l x_2||a_l x_3||a_l x_4| - |x_3 x_4 x_1||a_l x_3||a_l x_4||a_l x_1| \\ & + |x_4 x_1 x_2||a_l x_4||a_l x_1||a_l x_2| - |x_1 x_2 x_3||a_l x_1||a_l x_2||a_l x_3| \end{aligned}$$

and $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ are obtained by changing l into m, n and p respectively. We have then
$$\mathfrak{A} = \mathfrak{B} = \mathfrak{C} = \mathfrak{D} = |x_1 x_2 x_3 x_4| = \text{say } \Delta.$$
Write for convenience
$$|x_2 x_3 x_4||a_q x_1||a_r x_1||a_s x_1||a_t x_1||a_k x_2||a_k x_3||a_k x_4| = \Gamma_1,$$
then this becomes
$$\begin{aligned} \frac{\Delta \Gamma_1}{|x_1 x_2 x_3 x_4|^2} \{ & |a_m a_n a_p||a_m x_1||a_n x_1||a_p x_1| - |a_n a_p a_l||a_n x_1||a_p x_1||a_l x_1| \\ & + |a_p a_l a_m||a_p x_1||a_l x_1||a_m x_1| - |a_l a_m a_n||a_l x_1||a_m x_1||a_n x_1| \}. \end{aligned}$$

The term in the $\{\}$ is easily seen to be equal to $|a_l a_m a_n a_p|$. Equation 17 thus takes the form,

$$18. \qquad \begin{aligned} & |a_m a_n a_p| P^2_{kl} - |a_n a_p a_l| P^2_{km} + |a_p a_l a_m| P^2_{kn} - |a_l a_m a_n| P^2_{kp} = \\ & \qquad \frac{|a_l a_m a_n a_p|}{|x_1 x_2 x_3 x_4|} [\Gamma_1 - \Gamma_2 + \Gamma_3 - \Gamma_4]. \end{aligned}$$

Remembering to pay attention to the signs, we may write the term in [ ] as $\Sigma\Gamma$; then writing

19. $$\Sigma a_q = \lambda, \ \Sigma a_q a_r = \mu, \ \Sigma a_q a_r a_s = \nu, \ \Sigma a_q a_r a_s a_t = \pi,$$

we have

20. $$\Sigma\Gamma = \Sigma\,|x_2 x_3 x_4|\,(\pi - x_1\nu + x_1^2\mu - x_1^3\lambda + x_1^4)\big(a_k^3 - a_k^2(x_2 + x_3 + x_4) \\ + a_k(x_2x_3 + x_4x_2 + x_3x_4) - x_2x_3x_4\big).$$

Now writing as above $\theta = -|x_1x_2x_3x_4|$

and introducing the abbreviations α, β, γ, δ and λ, μ, ν, π, it is not difficult to see that

21. $$\theta = |x_1x_2x_3x_4|\,\{a_k^4 - a_k^3\alpha + a_k^2\beta - a_k\gamma + \delta - a_k^4 + a_k^3\lambda - a_k^2\mu + a_k\nu - \pi\};$$

referring now to equation 9 we have

22. $$\theta = |x_1x_2x_3x_4|\,\{P_k^2 - |a_ka_p||a_ka_q||a_ka_r||a_ka_s||a_ka_t|\,P_0^2\}.$$

Expanding equation 18 it becomes

$$\begin{aligned} & a_k^3\{\pi\Sigma\,|x_1x_2x_3| - \nu\Sigma x_1|x_2x_3x_4| + \mu\Sigma x_1^2|x_2x_3x_4| - \lambda\Sigma x_1^3|x_2x_3x_4| + \Sigma x_1^4|x_2x_3x_4|\} \\ & - a_k^2\{\pi\Sigma(x_2+x_3+x_4)|x_2x_3x_4| - \nu\Sigma x_1(x_2+x_3+x_4)|x_2x_3x_4| + \mu\Sigma x_1^2(x_2+x_3+x_4)|x_2x_3x_4| \\ & \quad - \lambda\Sigma x_1^3(x_2+x_3+x_4)|x_2x_3x_4| + \Sigma x_1^4(x_2+x_3+x_4)|x_2x_3x_4|\} \\ & \quad + a_k\{\pi\Sigma(x_2x_3+x_4x_2+x_3x_4)|x_2x_3x_4| - \nu\Sigma x_1(x_2x_3+x_4x_2+x_3x_4)|x_2x_3x_4| \\ & \quad + \mu\Sigma x_1^2(x_2x_3+x_4x_2+x_3x_4)|x_2x_3x_4| - \lambda\Sigma x_1^3(x_2x_3+x_4x_2+x_3x_4)|x_2x_3x_4| \\ & \quad + \Sigma x_1^4(x_2x_3+x_4x_2+x_3x_4)|x_2x_3x_4|\} - \{\pi\Sigma x_2x_3x_4|x_2x_3x_4| \\ & \quad - \nu x_1x_2x_3x_4\Sigma|x_2x_3x_4| + \mu x_1x_2x_3x_4\Sigma x_1|x_2x_3x_4| - \lambda x_1x_2x_3x_4\Sigma x_1^2|x_2x_3x_4| \\ & \quad + x_1x_2x_3x_4\Sigma x_1^3|x_2x_3x_4|\} = \\ & \qquad |a_ma_na_p|\,P_{kl}^2 - |a_na_pa_l|\,P_{km}^2 + |a_pa_la_m|\,P_{kn}^2 - |a_la_ma_n|\,P_{kp}^2. \end{aligned}$$

Of course in all these summations particular care must be taken to give the right signs to each term; for example, $\Sigma|x_2x_3x_4|$ means

$$|x_2x_3x_4| - |x_3x_4x_1| + |x_4x_1x_2| - |x_1x_2x_3|.$$

Using now equations 19 to 21 inclusive, we have, after simple reductions, for the reduced form of equation 18,

23. $$|a_ma_na_p|\,P_{kl}^2 - |a_na_pa_l|\,P_{km}^2 + |a_pa_la_m|\,P_{kn}^2 - |a_la_ma_n|\,P_{kp}^2 = \\ |a_la_ma_na_p|\,\{P_k^2 - |a_ka_p||a_ka_q||a_ka_r||a_ka_s||a_ka_t|\,P_0^2\}.$$

The factor $P_0^2 = 1$ being introduced simply for the sake of symmetry.

Now advance all the letters after k, that is, change l, m, n, p, q, r, s, t into m, n, p, q, r, s, t, l, and 23 becomes

24. $$|a_na_pa_q|\,P_{km}^2 - |a_pa_qa_m|\,P_{kn}^2 + |a_qa_ma_n|\,P_{kp}^2 - |a_ma_na_p|\,P_{kq}^2 = \\ |a_ma_na_pa_q|\,\{P_k^2 - |a_ka_q||a_ka_r||a_ka_s||a_ka_t||a_ka_l|\,P_0^2\}.$$

The coefficients of P_0^2 in 23 and 24 are respectively

$$[(a_m-a_n)(a_m-a_p)(a_n-a_p)(a_k-a_q)(a_k-a_r)(a_k-a_s)(a_k-a_t)]$$
$$(a_l-a_m)(a_l-a_n)(a_l-a_p)(a_k-a_p)$$

and
$$[(a_m-a_n)(a_m-a_p)(a_n-a_p)(a_k-a_q)(a_k-a_r)(a_k-a_s)(a_k-a_t)]$$
$$(a_m-a_q)(a_n-a_q)(a_p-a_q)(a_k-a_l).$$

Multiplying then 23 by the factor

$$(a_m-a_q)(a_n-a_q)(a_p-a_q)(a_k-a_l),$$

and 24 by the factor $(a_l-a_m)(a_l-a_n)(a_l-a_p)(a_k-a_p)$,
and subtracting one result from the other we eliminate P_0 and have

$$\begin{aligned}
&P_{kl}^2\,|a_ma_na_pa_q||a_ka_l|\\
&-P_{km}^2\{\,|a_na_pa_l||a_ma_q||a_na_q||a_pa_q||a_ka_l|+|a_na_pa_q||a_la_m||a_la_n||a_la_p||a_ka_p|\,\}\\
&+P_{kn}^2\{\,|a_pa_la_m||a_ma_q||a_na_q||a_pa_q||a_ka_l|+|a_pa_qa_m||a_la_m||a_la_n||a_la_p||a_ka_p|\,\}\\
&-P_{kp}^2\{\,|a_la_ma_n||a_ma_q||a_na_q||a_pa_q||a_ka_l|+|a_qa_ma_n||a_la_m||a_la_n||a_la_p||a_ka_p|\,\}\\
&+P_{kq}^2\{\,|a_ma_na_p||a_la_m||a_la_n||a_la_p||a_ka_p|\,\}=\\
&P_k^2\{a_la_ma_na_p||a_ma_q||a_na_q||a_pa_q||a_ka_l|-|a_ma_na_pa_q||a_la_m||a_la_n||a_la_p||a_ka_p|\,\}
\end{aligned}$$

This reduces to

$$\begin{aligned}
&P_{kl}^2\,|a_ma_na_pa_q||a_ka_l|-P_{km}^2\,|a_na_pa_qa_l||a_ka_m|+P_{kn}^2\,|a_pa_qa_la_m||a_ka_n|\\
\text{25.}\quad&-P_{kp}^2\,|a_qa_la_ma_n||a_ka_p|+P_{kq}^2\,|a_la_ma_na_p||a_ka_q|=\\
&P_k^2\,|a_la_ma_na_pa_q|.
\end{aligned}$$

If we here make again the substitution

$$\left|\begin{matrix} l & m & n & p & q & r & s & t\\ m & n & p & q & r & s & t & l\end{matrix}\right|$$

we get a new relation connecting the squares of P_k, P_{km}, P_{kn}, P_{kp}, P_{kq}, P_{kr}. The coefficient of P_k^2 will be $|a_ma_na_pa_qa_r|$:
multiplying this equation then by the factor

$$|a_la_m||a_la_n||a_la_p||a_la_q|,$$

and multiplying 25 by $|a_ma_r||a_na_r||a_pa_r||a_qa_r|$,
and subtracting one result from the other, we eliminate P_k and have a linear relation between the squares of

$$P_{kl},\ P_{km},\ P_{kn},\ P_{kp},\ P_{kq},\ P_{kr},\ \text{viz.}$$

$$\begin{aligned}
&P_{kl}^2\,|a_ma_na_pa_qa_r||a_ka_l|-P_{km}^2\,|a_na_pa_qa_ra_l||a_ka_m|\\
\text{26.}\quad&+P_{kn}^2\,|a_pa_qa_ra_la_m||a_ka_n|-P_{kp}\,|a_qa_ra_la_ma_n||a_ka_p|\\
&+P_{kq}^2\,|a_ra_la_ma_na_p||a_ka_q|-P_{kr}\,|a_la_ma_na_pa_q||a_ka_r|.
\end{aligned}$$

It is obvious that we might have eliminated P_k^2 between 23 and 24, and so have found a linear relation connecting the squares of

$$P_0,\ P_{kl},\ P_{km},\ P_{kn},\ P_{kp},\ P_{kq},$$

and by making the above substitution and eliminating P_0^2 between the two equations thus formed we would again arrive at 26. It is then clear that the functions with the indices

$$0,\ k,\ kl,\ km,\ kn,\ kp,\ kq,\ kr,\ ks,\ kt,$$

form a group of ten, such that any five being selected the squares of any of the remaining five can be expressed as a linear function of the squares of the chosen five. There are of course in all nine such groups, and these may be tabulated as follows:

0	*k*	*kl*	*km*	*kn*	*kp*	*kq*	*kr*	*ks*	*kt*
0	*l*	*lk*	*lm*	*ln*	*lp*	*lq*	*lr*	*ls*	*lt*
0	*m*	*mk*	*ml*	*mn*	*mp*	*mq*	*mr*	*ms*	*mt*
0	*n*	*nk*	*nl*	*nm*	*np*	*nq*	*nr*	*ns*	*nt*
0	*p*	*pk*	*pl*	*pm*	*pn*	*pq*	*pr*	*ps*	*pt*
0	*q*	*qk*	*ql*	*qm*	*qn*	*qp*	*qr*	*qs*	*qt*
0	*r*	*rk*	*rl*	*rm*	*rn*	*rp*	*rq*	*rs*	*rt*
0	*s*	*sk*	*sl*	*sm*	*sn*	*sp*	*sq*	*sr*	*st*
0	*t*	*tk*	*tl*	*tm*	*tn*	*tp*	*tq*	*tr*	*ts*

and the groups will be called the *k*-group, the *l*-group, etc.

We will now take up the case of three indices, which, as will be seen, divides into two sub-cases, according to the choice of the index. The two sub-cases give rise to two tables, the first containing 36 groups and the second containing 84 groups. As the method of working out these groups by Brunel's method has already been sufficiently indicated, I shall, in what follows, leave out as much as possible the purely algebraical processes of reduction, as they now become very long and wholly uninteresting. Squaring the function P_{klm} we have

$$27.\quad P_{klm}^2 = \frac{1}{|x_1x_2x_3x_4|^2}\Big\{ |x_2x_3x_4|^2 R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_mx_2||a_mx_3||a_mx_4|}{|a_kx_1||a_lx_1||a_mx_1|} + \ldots$$
$$- 2|x_2x_3x_4|\sqrt{R(x_1)R(x_2)}\,|a_kx_3||a_kx_4||a_lx_3||a_lx_4||a_mx_3||a_mx_4| + \ldots - \ldots + \ldots\Big\}$$

The radicals $\sqrt{R(x_1)R(x_2)}$, etc., may be eliminated between any six equations of the form 27, or between five equations of this form, having each a common index, say k, or between four equations having each two indices, say k and l, common. Choose multipliers A, B, C and D, such that the coefficient of $\sqrt{R(x_1)R(x_2)}$ (and in consequence the coefficients of all the other radicals) shall be zero in the sum $AP_{klm}^2 + BP_{kln}^2 + CP_{klp}^2 + DP_{kl}^2$.

This coefficient is easily seen to be

$$A|a_kx_3||a_kx_4||a_lx_3||a_lx_4||a_mx_3||a_mx_4| + B|a_kx_3||a_kx_4||a_lx_3||a_lx_4||a_nx_3||a_nx_4|$$
$$+ C|a_kx_3||a_kx_4||a_lx_3||a_lx_4||a_px_3||a_px_4| + D|a_kx_3||a_kx_4||a_lx_3||a_lx_4|$$

Striking out the common factor

$$|a_k x_3||a_k x_4||a_l x_3||a_l x_4|$$

the condition to be satisfied is

$$A\,|a_m x_3||a_m x_4| + B\,|a_n x_3||a_n x_4| + C\,|a_p x_3||a_p x_4| + D = 0\,; \tag{28.}$$

this is equivalent to

$$\begin{aligned} A + B + C &= 0\,, \\ a_m A + a_n B + a_p C &= 0\,, \\ a_m^2 A + a_n^2 B + a_p^2 C + D &= 0\,. \end{aligned}$$

These are easily seen to be satisfied by the values

$$A = |a_n a_p|\,,\quad B = |a_p a_m|\,,\quad C = |a_m a_n|\,,\quad D = -\,|a_m a_n a_p|\,. \tag{29.}$$

Introducing then these values of A, B, C and D, we have

$$\begin{aligned} &|a_n a_p|\,P^2_{klm} + |a_p a_m|\,P^2_{kln} + |a_m a_n|\,P^2_{klp} - |a_m a_n a_p|\,P^2_{kl} = \\ &\frac{1}{|x_1x_2x_3x_4|^2}\Big\{\,|a_n a_p|\Big[\,|x_2x_3x_4|^2 R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_mx_2||a_mx_3||a_mx_4|}{|a_kx_1||a_lx_1||a_mx_1|} + \ldots\Big] \\ &\quad + |a_p a_m|\Big[\,|x_2x_3x_4|^2 R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_nx_2||a_nx_3||a_nx_4|}{|a_kx_1||a_lx_1||a_nx_1|} + \ldots\Big] \\ &\quad + |a_m a_n|\Big[\,|x_2x_3x_4|^2 R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_px_2||a_px_3||a_px_4|}{|a_kx_1||a_lx_1||a_px_1|} + \ldots\Big] \\ &\quad - |a_m a_n a_p|\Big[\,|x_2x_3x_4|^2 R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4|}{|a_kx_1||a_lx_1|} + \ldots\Big]\Big\} \end{aligned} \tag{30.}$$

This is to be reduced just as in the case of two indices, viz. expand 30, so that the first line becomes

$$\begin{aligned} &\{\,|a_n a_p||x_2x_3x_4||a_nx_1||a_px_1|\ldots|a_tx_1||a_kx_2||a_kx_3||a_kx_4||a_lx_2|\ldots|a_lx_4|\,\} \\ &\times\{\,|x_2x_3x_4||a_mx_2||a_mx_3||a_mx_4| - |x_3x_4x_1||a_mx_3||a_mx_4||a_mx_1| + \ldots \\ &\qquad\qquad - |x_1x_2x_3||a_mx_1||a_mx_2||a_mx_3|\,\} + \ldots \end{aligned} \tag{31.}$$

There are three more terms similar to this to be obtained by simply advancing certain of the subscripts. The remaining three lines in 30 are to be expanded in a similar manner, and then the terms which have been introduced will disappear by aid of equations 28 and 29. The right-hand side of 30 is now easily reduced by aid of the following identities:

$$\begin{aligned} &|x_2x_3x_4| - |x_3x_4x_1| + |x_4x_1x_2| - |x_1x_2x_3| = 0\,, \\ &(x_2 + x_3 + x_4)\,|x_2x_3x_4| - (x_3 + x_4 + x_1)\,|x_3x_4x_1| \\ &\quad + (x_4 + x_1 + x_2)\,|x_4x_1x_2| - (x_1 + x_2 + x_3)\,|x_1x_2x_3| = 0\,, \\ &(x_2x_3 + x_3x_4 + x_4x_2)\,|x_2x_3x_4| - (x_3x_4 + x_4x_1 + x_1x_3)\,|x_3x_4x_1| \\ &\quad + (x_4x_1 + x_1x_2 + x_2x_4)\,|x_4x_1x_2| - (x_1x_2 + x_2x_3 + x_3x_1)\,|x_1x_2x_3| = 0\,, \\ &x_2x_3x_4\,|x_2x_3x_4| - x_3x_4x_1\,|x_3x_4x_1| + x_4x_1x_2\,|x_4x_1x_2| - x_1x_2x_3\,|x_1x_2x_3| = -\,|x_1x_2x_3x_4|\,. \end{aligned} \tag{32.}$$

This last equation written out in full is

33. $$\begin{vmatrix} x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = - \begin{vmatrix} x_2x_3x_4 & x_3x_4x_1 & x_4x_1x_2 & x_1x_2x_3 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

which is clearly true, as each of the determinants is equal to the product of differences of the x's.

Another form of expressing $|x_1x_2x_3x_4|$ is obviously

34. $$\begin{vmatrix} x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = - x_1^3x_2^3x_3^3x_4^3 \begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{x_1} & \frac{1}{x_2} & \frac{1}{x_3} & \frac{1}{x_4} \\ \frac{1}{x_1^2} & \frac{1}{x_2^2} & \frac{1}{x_3^2} & \frac{1}{x_4^2} \\ \frac{1}{x_1^3} & \frac{1}{x_2^3} & \frac{1}{x_3^3} & \frac{1}{x_4^3} \end{vmatrix}$$

For the general case of n variables $x_1, x_2 \ldots x_n$, it is not difficult to see that $|x_1, x_2 \ldots x_n|$ satisfies the identity

35. $$\begin{vmatrix} x_1^{n-1}, & x_2^{n-1} & \ldots & x_n^{n-1} \\ x_1^{n-2}, & x_2^{n-2} & \ldots & x_n^{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ x_1, & x_2, & \ldots & x_n \\ 1 & 1 & \ldots & 1 \end{vmatrix} = (-1)^{n-1} \begin{vmatrix} x_2x_3\ldots x_n, & x_3x_4\ldots x_nx_1 & \ldots, & x_1x_2\ldots x_{n-1} \\ x_1^{n-2}, & x_2^{n-2} & \ldots\ldots, & x_n^{n-2} \\ x_1^{n-3}, & x_2^{n-3} & \ldots\ldots, & x_n^{n-3} \\ \cdots & \cdots & \cdots & \cdots \\ x_1, & x_2 & \ldots\ldots & x_n \\ 1 & 1 & \ldots\ldots & 1 \end{vmatrix}$$

or

36. $$\begin{vmatrix} x_1^{n-1}, & x_2^{n-1} & \ldots & x_n^{n-1} \\ x_1^{n-2}, & x_2^{n-2} & \ldots & x_n^{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ x_1 & x_2 & \ldots & x_n \\ 1 & 1 & \ldots & 1 \end{vmatrix} = (-)^{n-1} [x_1x_2\ldots x_n]^{n-1} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \frac{1}{x_1} & \frac{1}{x_2} & \ldots & \frac{1}{x_n} \\ \frac{1}{x_1^2} & \frac{1}{x_2^2} & \ldots & \frac{1}{x_n^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{x_1^{n-1}} & \frac{1}{x_2^{n-1}} & \ldots & \frac{1}{x_n^{n-1}} \end{vmatrix}.$$

The following identities are also very important in the sequence, viz. writing

37. $A = |a_na_pa_q|$, $B = -|a_pa_qa_m|$, $C = |a_qa_ma_n|$, $D = -|a_ma_na_p|$,

we have $$A + B + C + D = 0,$$

38. $$a_mA + a_nB + a_pC + a_qD = 0,$$
$$a_m^2A + a_n^2B + a_p^2C + a_q^2D = 0.$$

A similar group of identities may easily be written down for the general case. Using these last identities and taking the four functions

$$P^2_{klm},\ P^2_{kln},\ P^2_{klp},\ P^2_{klq},$$

we can, by multiplying the first by A, the second by B, etc., and taking the sum, eliminate the radicals $\sqrt{R(x_1)R(x_2)}$, $\sqrt{R(x_1)R(x_3)}$, etc. Forming this sum it is only necessary to show that

39. $$A\,|a_m x_3||a_m x_4| + B\,|a_n x_3||a_n x_4| + C\,|a_p x_3||a_p x_4| + D\,|a_q x_3||a_q x_4| = 0;$$

and this equation of condition is at once seen to be satisfied by the values

39′. $$A = |a_n a_p a_q|,\ B = -|a_p a_q a_m|,\ C = |a_q a_m a_n|,\ D = -|a_m a_n a_p|.$$

We have now

$$\begin{aligned}
&|a_n a_p a_q|\,P^2_{klm} - |a_p a_q a_m|\,P^2_{kln} + |a_q a_m a_n|\,P^2_{klp} - |a_m a_n a_p|\,P^2_{klq} = \\
&\frac{1}{|x_1x_2x_3x_4|^2}\Bigg\{ |a_n a_p a_q|\Big[\,|x_2x_3x_4|^2 R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_mx_2||a_mx_3||a_mx_4|}{|a_kx_1||a_lx_1||a_mx_1|} + \ldots\Big] \\
\text{40.}\quad &- |a_p a_q a_m|\Big[\,|x_2x_3x_4|^2 R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_nx_2||a_nx_3||a_nx_4|}{|a_kx_1||a_lx_1||a_nx_1|} + \ldots\Big] \\
&+ |a_q a_m a_n|\Big[\,|x_2x_3x_4|^2 R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_px_2||a_px_3||a_px_4|}{|a_kx_1||a_lx_1||a_px_1|} + \ldots\Big] \\
&- |a_m a_n a_p|\Big[\,|x_2x_3x_4|^2 R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_qx_2||a_qx_3||a_qx_4|}{|a_kx_1||a_lx_1||a_qx_1|} + \ldots\Big]\Bigg\}.
\end{aligned}$$

This may be briefly written in the form

$$\begin{aligned}
&|a_n a_p a_q|\,P^2_{klm} - |a_p a_q a_m|\,P^2_{kln} + |a_q a_m a_n|\,P^2_{klp} - |a_m a_n a_p|\,P^2_{klq} = \\
&\frac{1}{|x_1x_2x_3x_4|^2}\Bigg\{ |a_n a_p a_q|\Big[\,|x_2x_3x_4|^2 R(x_1)\frac{[a_k,\ a_l,\ a_m][x_2,\ x_3,\ x_4]}{[a_k,\ a_l,\ a_m][x_1]} + \ldots\Big] \\
\text{41.}\quad &- |a_p a_q a_m|\Big[\,|x_2x_3x_4|^2 R(x_1)\frac{[a_k,\ a_l,\ a_n][x_2,\ x_3,\ x_4]}{[a_k,\ a_l,\ a_n][x_1]} + \ldots\Big] \\
&+ |a_q a_m a_n|\Big[\,|x_2x_3x_4|^2 R(x_1)\frac{[a_k,\ a_l,\ a_p][x_2,\ x_3,\ x_4]}{[a_k,\ a_l,\ a_p][x_1]} + \ldots\Big] \\
&- |a_m a_n a_p|\Big[\,|x_2x_3x_4|^2 R(x_1)\frac{[a_k,\ a_l,\ a_q][x_2,\ x_3,\ x_4]}{[a_k,\ a_l,\ a_q][x_1]} + \ldots\Big]\Bigg\}
\end{aligned}$$

Expanding this just as in the case of two indices and the case of equation 30, we have for the first line on the right-hand side of the equation

$$\begin{aligned}
&\{\,|a_n a_p a_q||x_2x_3x_4|\,[a_n,\ a_p,\ a_q,\ a_r,\ a_s,\ a_t][x_1][a_k,\ a_l][x_2,\ x_3,\ x_4]\,\} \\
\times &\{\,|x_2x_3x_4|\,[a_m][x_2,\ x_3,\ x_4| - |x_3x_4x_1|\,[a_m][x_3,\ x_4,\ x_1] \\
&\qquad + |x_4x_1x_2|\,[a_m][x_4,\ x_1,\ x_2] - |x_1x_2x_3|\,[a_m][x_1,\ x_2,\ x_3]\,\}
\end{aligned}$$

+ three similar terms.

The remaining three lines of 41 are to be expanded in the same manner, and then it will at once be seen that the extra terms which have been introduced will vanish on account of the relations 39 and 39′. Consider now the terms containing the factor $R(x_1)$: they have obviously the common factor

$$|x_2x_3x_4|\,[a_r,\,a_s,\,a_t][x_1]\,.\,[a_k,\,a_l][x_2,\,x_3,\,x_4]$$

and the remaining factor is

$$\begin{aligned}&|a_na_pa_q|\,[a_n,\,a_p,\,a_q][x_1]-|a_pa_qa_m|\,[a_p,\,a_q,\,a_m][x_1]\\&+|a_qa_ma_n|\,[a_q,\,a_m,\,a_n][x_1]-|a_ma_na_p|\,[a_m,\,a_n,\,a_p][x_1]=,\text{ say }K.\end{aligned}$$

Expanding K and using the identities 32 and 33, we have

42. $$K=-|a_ma_na_pa_q|\,.$$

The first line on the right-hand side of equations 40 or 41 contains four terms, the first of which contains the factor already mentioned, viz.

43. $$\begin{aligned}|x_2x_3x_4|\,[a_m][x_2,\,x_3,\,x_4]-|x_3x_4x_1|\,[a_m][x_3,\,x_4,\,x_1]+|x_4x_1x_2|\,[a_m][x_4,\,x_1,\,x_2]&\\-|x_1x_2x_3|\,[a_m][x_1,\,x_2,\,x_3]&;\end{aligned}$$

the factor in each of the remaining terms is derived from this by changing m into n, p, q respectively. The same is true for the remaining three lines on the right-hand side of 40 or 41. This factor is independent of a_m, and the others not written down are equally independent of n, p, or q; for writing 43 out in full it is

$$\begin{aligned}&|x_2x_3x_4||a_mx_2||a_mx_3||a_mx_4|-|x_3x_4x_1||a_mx_3||a_mx_4||a_mx_1|\\&+|x_4x_1x_2||a_mx_4||a_mx_1||a_mx_2|-|x_1x_2x_3||a_mx_1||a_mx_2||a_mx_3|\end{aligned}$$

and this is equal to

$$\begin{aligned}&a_m^3\Sigma\,|x_2x_3x_4|+a_m^2\Sigma\,(x_2+x_3+x_4)\,|x_2x_3x_4|\\&\qquad+a_m\Sigma\,(x_2x_3+x_3x_4+x_4x_2)\,|x_2x_3x_4|+\Sigma x_2x_3x_4\,|x_2x_3x_4|\,.\end{aligned}$$

The first three terms of this vanish by virtue of the identities 32, and the fourth term by 33 becomes $=-|x_1x_2x_3x_4|$.

The right-hand side of 40 and 41 thus contains the factor

$$-|a_ma_na_pa_q|\,.-|x_1x_2x_3x_4|=|a_ma_na_pa_q||x_1x_2x_3x_4|\,.$$

The right-hand member of 41 takes now the form

$$\frac{1}{|x_1x_2x_3x_4|}\,|a_ma_na_pa_q|\,\Sigma\,|x_2x_3x_4|\,[a_r,\,a_s,\,a_t][x_1][a_k,\,a_l][x_2,\,x_3,\,x_4]\,.$$

The Σ of course refers only to the cyclic permutations of the suffixes 1, 2, 3, 4. We have now to determine the value of the quantity under the summation sign, viz.

$$\Sigma\,|x_2x_3x_4||a_rx_1||a_sx_1||a_tx_1||a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4|$$

in order to find the relation connecting the squares of P_{klm}, P_{kln}, P_{klp}, P_{klq}. For greater generality and completeness, however, it is better to go back to equation 30 and reduce it, *i. e.* find the linear relation connecting the squares of P_{klm}, P_{kln}, P_{klp}, P_{kl}. It will then be seen that by making the substitution

$$\left|\begin{matrix}k & l & m & n & p & q & r & s & t\\ l & m & n & p & q & r & s & t & k\end{matrix}\right|$$

and eliminating one quantity, we arrive at what we might obtain directly by completing the reduction of equation 40.

The second factor in 31 is easily seen to be

$$44. \qquad = - \vert x_1x_2x_3x_4 \vert ,$$

and the same is true for the corresponding factors in the remaining three lines of 30. Now adding together the first factors of the four lines in 30, viz. those similar to the first factor in 31, we have

$$\begin{aligned} &\vert a_na_p\vert\vert x_2x_3x_4\vert [a_n, a_p \ldots a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4] \\ +&\vert a_pa_m\vert\vert x_2x_3x_4\vert [a_p, a_q \ldots a_t, a_m][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4] \\ +&\vert a_ma_n\vert\vert x_2x_3x_4\vert [a_m, a_p \ldots a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4] . \end{aligned}$$

The fourth line need not be written down, as its second factor is zero.

Adding these terms we have

$$\begin{aligned} &\Sigma \{ \vert x_2x_3x_4\vert [a_q, a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4] \\ &\times \vert a_na_p\vert\vert a_nx_1\vert\vert a_px_1\vert + \vert a_pa_m\vert\vert a_px_1\vert\vert a_mx_1\vert + \vert a_ma_n\vert\vert a_mx_1\vert\vert a_nx_1\vert \}. \end{aligned}$$

Now $\vert a_na_p\vert + \vert a_pa_m\vert + \vert a_ma_n\vert = 0,$

$$(a_n + a_p)\vert a_na_p\vert + (a_p + a_m)\vert a_pa_n\vert + (a_m + a_n)\vert a_ma_n\vert = 0 ,$$

and $a_na_p\vert a_na_p\vert + a_pa_m\vert a_pa_m\vert + a_ma_n\vert a_ma_n\vert = \vert a_ma_na_p\vert ,$

so that the above reduces to

$$45. \qquad \vert a_ma_na_p\vert \Sigma \vert x_2x_3x_4\vert [a_q, a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4],$$

the summation referring to the subscripts 1, 2, 3, 4. Equation 30, or equation 31, becomes now, by taking into acccount 44 and 45,

$$\begin{aligned} &\vert a_na_p\vert P^2_{klm} + \vert a_pa_m\vert P^2_{kln} + \vert a_ma_n\vert P^2_{klp} - \vert a_ma_na_p\vert P^2_{kl} = \\ 46. \quad &\frac{1}{\vert x_1x_2x_3x_4\vert} \vert a_ma_na_p\vert \Sigma \vert x_2x_3x_4\vert [a_q, a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4] . \end{aligned}$$

For brevity write as before

$$\Sigma a_q = \lambda,\ \Sigma a_qa_r = \mu,\ \Sigma a_qa_ra_s = \nu,\ a_qa_ra_sa_t = \pi,$$

the summations extending over the subscripts q, r, s, t. We have now

$$\begin{aligned} &\Sigma \vert x_2x_3x_4\vert [a_q, a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4] \\ &= \Sigma \vert x_2x_3x_4\vert (\pi - x_1\nu + x_1^2\mu - x_1^3\lambda + x_1^4)\{(a_k^3 - a_k^2(x_2+x_3+x_4) + a_k(x_2x_3+x_3x_4+x_4x_2) - x_2x_3x_4) \\ &\qquad \times (a_l^3 - a_l^2(x_2 + x_3 + x_4) + a_l(x_2x_3 + x_3x_4 + x_4x_2) - x_2x_3x_4)\} \\ &= \Sigma \vert x_2x_3x_4\vert (\pi - x_1\nu + x_1^2\mu - x_1^3\lambda + x_1^4)\{a_k^3a_l^3 - a_k^2a_l^2(a_k + a_l)(x_2 + x_3 + x_4) \\ &\qquad + a_k^2a_l^2(x_2 + x_3 + x_4)^2 + a_ka_l(a_k^2 + a_l^2)(x_2x_3 + x_3x_4 + x_4x_2) \\ 47. \quad &\qquad - a_ka_l(a_k + a_l)(x_2 + x_3 + x_4)(x_2x_3 + x_3x_4 + x_4x_2) + a_ka_l(x_2x_3 + x_3x_4 + x_4x_2)^2 \\ &\qquad + (a_k^2 + a_l^2)(x_2 + x_3 + x_4)x_2x_3x_4 - (a_k + a_l)(x_2x_3 + x_3x_4 + x_4x_2)x_2x_3x_4 \\ &\qquad - (a_k^3 + a_l^3)x_2x_3x_4 + x_2^2x_3^2x_4^2\}. \end{aligned}$$

The identities given above and a number of similar ones serve for the reduction of this. Paying attention to the signs we have, for a few of the identities, the following:

$$\begin{aligned}\Sigma x_1^3\,|x_2x_3x_4| &= -\theta\\ \Sigma(x_2^2+x_3^2+x_4^2)\,|x_2x_3x_4| &= +\theta\\ \Sigma x_2x_3x_4\,|x_2x_3x_4| &= -\theta\\ \Sigma x_1(x_2x_3+x_3x_4+x_4x_2)\,|x_2x_3x_4| &= +\theta\\ \Sigma x_1^4\,|x_2x_3x_4| &= -\theta\alpha\\ \Sigma x_1^3(x_2+x_3+x_4)\,|x_2x_3x_4| &= 0\\ \Sigma x_1^2(x_2^2+x_3^2+x_4^2)\,|x_2x_3x_4| &= \theta\alpha, \text{ etc., etc., etc.}\end{aligned}$$

It is easy to write down any number of these.

In the expansion of 47 the term containing the factor π is, on multiplying both sides of the equation by $|a_k, a_l|$

$$\begin{aligned}\pi\,|a_ka_l|\,\{a_k^3a_l^3\Sigma\,|x_2x_3x_4| &- a_k^2a_l^2(a_k+a_l)\Sigma(x_2+x_3+x_4)\,|x_2x_3x_4|\\ &- a_k^2a_l^2\Sigma(x_2+x_3+x_4)^2\,|x_2x_3x_4|\\ &+ a_ka_l(a_k^2+a_l^2)\Sigma(x_2x_3+x_3x_4+x_4x_2)\,|x_2x_3x_4|\\ &- a_ka_l(a_k+a_l)\Sigma(x_2+x_3+x_4)(x_2x_3+x_3x_4+x_4x_2)\,|x_2x_3x_4|\\ &- a_ka_l\Sigma(x_2x_3+x_3x_4+x_4x_2)^2\,|x_2x_3x_4|\\ -(a_k^2+a_l^2)\Sigma\,x_2x_3x_4(x_2+x_3+x_4)\,|x_2x_3x_4| &-(a_k+a_l)\Sigma\,x_2x_3x_4(x_2x_3+x_3x_4+x_4x_2)\,|x_2x_3x_4|\\ -(a_k^3+a_l^3)\Sigma\,x_2x_3x_4\,|x_2x_3x_4| &+ \Sigma\,x_2^2x_3^2x_4^2\,|x_2x_3x_4|\,\}.\end{aligned}$$

Now we have $\Sigma\,|x_2x_3x_4| = 0$, $\Sigma(x_2+x_3+x_4)\,|x_2x_3x_4| = 0$,

$\Sigma(x_2+x_3+x_4)^2\,|x_2x_3x_4| = \Sigma(x_2^2+x_3^2+x_4^2)\,|x_2x_3x_4| + 2\Sigma(x_2x_3+x_3x_4+x_4x_2\,|x_2x_3x_4|$

it is easy to see that $\Sigma(x_2^2+x_3^2+x_4^2)\,|x_2x_3x_4| = 0$, and we know that

$$\begin{aligned}\Sigma(x_2x_3+x_3x_4+x_4x_2)\,|x_2x_3x_4| &= 0,\\ \therefore\quad \Sigma(x_2+x_3+x_4)^2\,|x_2x_3x_4| &= 0,\end{aligned}$$

also

$$\begin{aligned}\Sigma(x_2x_3+x_3x_4+x_4x_2)^2\,|x_2x_3x_4| &= 0,\\ \Sigma\,x_2x_3x_4\,|x_2x_3x_4| &= -\theta,\\ \Sigma\,x_2x_3x_4(x_2+x_3+x_4)\,|x_2x_3x_4| &= -\alpha\theta,\\ \Sigma x_2x_3x_4(x_2x_3+x_3x_4+x_4x_2)\,|x_2x_3x_4| &= -\beta\theta,\\ \Sigma x_2^2x_3^2x_4^2\,|x_2x_3x_4| &= \gamma\theta.\end{aligned}$$

We have now after easy reductions

$$\begin{aligned}&\pi\theta\{a_k^4-\alpha a_k^3+\beta a_k^2-\gamma a_k+\delta^*\}\\ -\,&\pi\theta\{a_l^4-\alpha a_l^3+\beta a_l^2-\gamma a_l+\delta^*\},\end{aligned}$$

the δ^* being introduced simply for symmetry. In the following the terms marked by the asterisk are all introduced simply for symmetry, and obviously cancel each other. Making reductions for the remaining terms similar to those just made for the term containing π we find

$$\begin{aligned}
&|a_ka_l|\Sigma|x_2x_3x_4|[a_q,\ a_r,\ a_s,\ a_t][x_1][a_k][x_2,\ x_3,\ x_4][a_l][x_2,\ x_3,\ x_4]\\
&=|x_1x_2x_3x_4|\{\pi(a_k^4-\alpha a_k^3+\beta a_k^2-\gamma a_k+\delta^*)\\
&\qquad-\pi(a_l^4-\alpha a_l^3+\beta a_l^2-\gamma a_l+\delta^*)\\
&\qquad-\nu a_l(a_k^4-\alpha a_k^3+\beta a_k^2-\gamma a_k^*+\delta)\\
&\qquad+\nu a_k(a_l^4-\alpha a_l^3+\beta a_l^2-\gamma a_l^*+\delta)\\
&\qquad+\mu a_l^2(a_k^4-\alpha a_k^3+\beta a_k^{2*}-\gamma a_k+\delta)\\
&\qquad-\mu a_k^2(a_l^4-\alpha a_l^3+\beta a_l^{2*}-\gamma a_l+\delta)\\
&\qquad-\lambda a_l^3(a_k^4-\alpha a_k^{3*}+\beta a_k^2-\gamma a_k+\delta)\\
&\qquad+\lambda a_k^3(a_l^4-\alpha a_l^{3*}+\beta a_l^2-\gamma a_l+\delta)\\
&\qquad+\ a_l^4(a_k^{4*}-\alpha a_k^3+\beta a_k^2-\gamma a_k+\delta)\\
&\qquad-\ a_k^4(a_l^{4*}-\alpha a_l^3+\beta a_l^2-\gamma a_l+\delta)\}
\end{aligned}$$

or

$$\begin{aligned}
&=|x_1x_2x_3x_4|\{(\pi-\nu a_l+\mu a_l^2-\lambda a_l^3+a_l^4)(a_k^4-\alpha a_k^3+\beta a_k^2-\gamma a_k+\delta)\\
&\qquad-(\pi-\nu a_k+\mu a_k^2-\lambda a_k^3+a_k^4)(a_l^4-\alpha a_l^3+\beta a_l^2-\gamma a_l+\delta)\}.
\end{aligned}$$

From equation 9 however we have

$$\begin{aligned}
a_k^4-\alpha a_k^3+\beta a_k^2-\gamma a_k+\delta&=P_k^2,\ =|a_kx_1||a_kx_2||a_kx_3||a_kx_4|\\
a_l^4-\alpha a_l^3+\beta a_l^2-\gamma a_l+\delta&=P_l^2,\ =|a_lx_1||a_lx_2||a_lx_3||a_lx_4|
\end{aligned}$$

similarly we have

$$\begin{aligned}
\pi-\nu a_l+\mu a_l^2-\lambda a_l^3+a_l^4&=|a_qa_l||a_ra_l||a_sa_l||a_ta_l|\\
\pi-\nu a_k+\mu a_k^2-\lambda a_k^3+a_k^4&=|a_qa_k||a_ra_k||a_sa_k||a_ta_k|
\end{aligned}$$

and therefore finally

48.
$$\begin{aligned}
&|a_ka_l|\{a_na_p|P_{klm}^2+|a_pa_m|P_{kln}^2+|a_ma_n|P_{klp}^2\}\\
&=|a_ma_na_p|\{\,|a_ka_l|P_{kl}^2+|a_qa_l||a_ra_l||a_sa_l||a_ta_l|P_k^2-|a_qa_k||a_ra_k||a_sa_k||a_ta_k|P_l^2\}.
\end{aligned}$$

Effect upon this the substitution

$$\left|\begin{matrix} m & n & p & q & r & s & t\\ n & p & q & r & s & t & m\end{matrix}\right|$$

and we arrive at a new relation

49.
$$\begin{aligned}
&|a_ka_l|\{\,|a_pa_q|P_{kln}^2+|a_qa_n|P_{klp}^2+|a_na_p|P_{klq}^2\}\\
&=|a_na_pa_q|\{\,|a_ka_l|P_{kl}^2+|a_ra_l||a_sa_l||a_ta_l||a_ma_l|P_k^2-|a_ra_k||a_sa_k||a_ta_k||a_ma_k|P_l^2\}.
\end{aligned}$$

Eliminating P_{kl}^2 between 48 and 49 and we find a linear relation connecting

$$P_{klm}^2,\ P_{kln}^2,\ P_{klp}^2,\ P_{klq}^2,\ P_{kl}^2,\ P_k^2,$$

viz., after dropping out a common factor $|a_na_p|$,

50.
$$\begin{aligned}
&|a_ka_l|\{\,|a_na_pa_q|P_{klm}^2-|a_pa_qa_m|P_{kln}^2+|a_qa_ma_n|P_{klp}^2-|a_ma_na_p|P_{klq}^2\}\\
&=|a_ma_na_pa_q|\{\,|a_ra_k||a_sa_k||a_ta_k|P_l^2-|a_ra_l||a_sa_l||a_ta_l|P_k^2\}.
\end{aligned}$$

If instead of eliminating P^2_{kl} between 48 and 49 we had eliminated P^2_l, we would arrive at a linear relation connecting

$$P^2_{klm},\ P^2_{kln},\ P^2_{klp},\ P^2_{klq},\ P^2_{kl},\ P^2_k,$$

say

51. $$AP^2_{klm} + BP^2_{kln} + CP^2_{klp} + DP^2_{klq} + EP^2_{kl} + FP^2_k = 0.$$

Now effect upon 50 the substitution

$$\left| \begin{matrix} m & n & p & q & r & s & t \\ n & p & q & r & s & t & m \end{matrix} \right| = \text{say } \Omega,$$

then we will obviously obtain a relation of the form,

52. $$A'P^2_{kln} + B'P^2_{klp} + C'P^2_{klq} + D'P_{klr} + E'P^2_{kl} + F'P^2_k = 0.$$

Eliminating P^2_{kl} between 51 and 52 and we have a linear relation connecting five of the P-functions possessing triple indices, and one possessing a single index, viz., a relation of the form

53. $$A''P^2_{klm} + B''P^2_{kln} + C''P^2_{klp} + D''P^2_{klq} + E''P^2_{klr} + F''P^2_k = 0.$$

Or, if P^2_k had been eliminated, we would have a relation connecting five P-functions with triple indices, and one with a double index, viz.

54. $$A'''P^2_{klm} + B'''P^2_{kln} + C'''P^2_{klp} + D''P^2_{klq} + E'''P^2_{klr} + F'''P^2_{kl} = 0.$$

Effecting the substitution Ω upon 54 and there results an equation of the form,

55. $$A^{\text{iv}}P^2_{kln} + B^{\text{iv}}P^2_{klp} + C^{\text{iv}}P^2_{klq} + D^{\text{iv}}P^2_{klr} + E^{\text{iv}}P^2_{kls} + F^{\text{iv}}P^2_{kl} = 0.$$

Now finally eliminating P^2_{kl} between 54 and 55 and we arrive at a linear relation connecting the squares of six P-functions with triple indices, viz.

56. $$A^{\text{v}}P^2_{klm} + B^{\text{v}}P^2_{kln} + C^{\text{v}}P^2_{klp} + D^{\text{v}}P^2_{klq} + E^{\text{v}}P^2_{klr} + F^{\text{v}}P^2_{kls} = 0.$$

Of course, the substitution Ω performed upon 56 would give

57. $$A^{\text{vi}}P^2_{kln} + B^{\text{vi}}P^2_{klp} + C^{\text{vi}}P^2_{klq} + D^{\text{vi}}P^2_{klr} + E^{\text{vi}}P^2_{kls} + F^{\text{vi}}P^2_{klt} = 0.$$

We arrive thus at the conclusion that the ten functions

$$P_k,\ P_l,\ P_{kl},\ P_{klm},\ P_{kln},\ P_{klp},\ P_{klq},\ P_{klr},\ P_{kls},\ P_{klt}$$

form a group such that selecting any five of them the square of any one of the remaining five is linearly expressible in terms of the squares of the chosen five. There are in all 36 such groups, and they are given in the following table:

k	*l*	*kl*	*klm*	*kln*	*klp*	*klq*	*klr*	*kls*	*klt*
k	*m*	*km*	*kml*	*kmn*	*kmp*	*kmq*	*kmr*	*kms*	*kmt*
k	*n*	*kn*	*knl*	*knm*	*knp*	*knq*	*knr*	*kns*	*knt*
k	*p*	*kp*	*kpl*	*kpm*	*kpn*	*kpq*	*kpr*	*kps*	*kpt*
k	*q*	*kq*	*kql*	*kqm*	*kqn*	*kqp*	*kqr*	*kqs*	*kqt*
k	*r*	*kr*	*krl*	*krm*	*krn*	*krp*	*krq*	*krs*	*krt*
k	*s*	*ks*	*ksl*	*ksm*	*ksn*	*ksp*	*ksq*	*ksr*	*kst*

k	t	kt	ktl	ktm	ktn	ktp	ktq	ktr	kts
l	m	lm	lmk	lmn	lmp	lmq	lmr	lms	lmt
l	n	ln	lnk	lnm	lnp	lnq	lnr	lns	lnt
l	p	lp	lpk	lpm	lpn	lpq	lpr	lps	lpt
l	q	lq	lqk	lqm	lqn	lqp	lqr	lqs	lqt
l	r	lr	lrk	lrm	lrn	lrp	lrq	lrs	lrt
l	s	ls	lsk	lsm	lsn	lsp	lsq	lsr	lst
l	t	lt	ltk	ltm	ltn	ltp	ltq	ltr	lts
m	n	mn	mnk	mnl	mnp	mnq	mnr	mns	mnt
m	p	mp	mpk	mpl	mpn	mpq	mpr	mps	mpt
m	q	mq	mqk	mql	mqn	mqp	mqr	mqs	mqt
m	r	mr	mrk	mrl	mrn	mrp	mrq	mrs	mrt
m	s	ms	msk	msl	msn	msp	msq	msr	mst
m	t	mt	mtk	mtl	mtn	mtp	mtq	mtr	mts
n	p	np	npk	npl	npm	npq	npr	nps	npt
n	q	nq	nqk	nql	nqm	nqp	nqr	nqs	nqt
n	r	nr	nrk	nrl	nrm	nrp	nrq	nrs	nrt
n	s	ns	nsk	nsl	nsm	nsp	nsq	nsr	nst
n	t	nt	ntk	ntl	ntm	ntp	ntq	ntr	nts
p	q	pq	pqk	pql	pqm	pqn	pqr	pqs	pqt
p	r	pr	prk	prl	prm	prn	prq	prs	prt
p	s	ps	psk	psl	psm	psn	psq	psr	pst
p	t	pt	ptk	ptl	ptm	ptn	ptq	ptr	pts
q	r	qr	qrk	qrl	qrm	qrn	qrp	qrs	qrt
q	s	qs	qsk	qsl	qsm	qsn	qsp	qsr	qst
q	t	qt	qtk	qtl	qtm	qtn	qtp	qtr	qts
r	s	rs	rsk	rsl	rsm	rsn	rsp	rsq	rst
r	t	rt	rtk	rtl	rtm	rtn	rtp	rtq	rts
s	t	st	stk	stl	stm	stn	stp	stq	str

Consider now the functions with four indices and find values of A, B, C, D, such that the radicals $\sqrt{R(x_1)R(x_2)}$, etc. shall vanish in the sum

$$AP^2_{klmn} - BP^2_{klmp} + CP^2_{klmq} - DP^2_{klmr}.$$

The coefficient of $\sqrt{R(x_1)R(x_2)}$ in this sum is, leaving out the common factor $|x_2x_3x_4||x_3x_4x_1|$,

$$A[a_k, a_l, a_m, a_n][x_3][a_k, a_l, a_m, a_n][x_4] - B[a_k, a_l, a_m, a_p][x_3][a_k, a_l, a_m, a_p][x_4]$$
$$+ C[a_k, a_l, a_m, a_q][x_3][a_k, a_l, a_m, a_q][x_4] - D[a_k, a_l, a_m, a_r][x_3][a_k, a_l, a_m, a_r][x_4] = 0.$$

Taking out the common factor

$$[a_k,\ a_l,\ a_m][x_3][a_k,\ a_l,\ a_m][x_4],$$

and this becomes

$$A\,|a_n x_3||a_n x_4| - B\,|a_p x_3||a_p x_4| + C\,|a_q x_3||a_q x_4| - D\,|a_r x_3||a_r x_4| = 0,$$

giving

$$A = |a_p a_q a_r|, \quad B = |a_q a_r a_n|,$$
$$C = |a_r a_n a_p|, \quad D = |a_n a_p a_q|.$$

Introducing here the values

$$R(x_1) = [a_k,\ a_l,\ a_m,\ a_n,\ a_p,\ a_q,\ a_r,\ a_s,\ a_t][x_1],\ \text{etc.}$$

we ought to be able to show that the squares of any six of the functions whose indices are *kl lm mk klm klmn klmp klmq klmr klms klmt* are connected by a linear relation. We would then have a table of 84 groups similar to the above, and such that the squares of any six functions in a given group are connected by a linear relation. In the case of quadruple indices there would also be a second table containing 126 groups, of which

klm lmn mnk nkl klmn pqrs qrst rstp stpq tpqr

is the first, and the squares of any six of these functions should also be connected by a linear relation.

We would thus have in all 256 groups giving linear relations between the squares of six P-functions. There would be 840 relations of this kind, but not all, of course, asyzygetic.

These 256 groups of ten functions each might be called the 256 decads, and they would obviously correspond to the 16 Kummer hexads in the case of the double theta-functions, viz. between the squares of any four theta-functions of a Kummer hexad there exists a linear relation, and between the squares of any six of the P-functions belonging to a given decad there exists a linear relation, the same kind of relation will obviously exist between the squares of the six corresponding quadruple theta-functions. We have thus hexads of double theta-functions of which the squares of any four are connected by a linear relation; octads of triple theta-functions of which the squares of any five are connected by a linear relation; decads of quadruple theta-functions of which the squares of any six are connected by a linear relation, and in general $2(p+1)$-ads of p-tuple theta-functions of which the squares of any $p+2$ are connected by a linear relation. It seems highly probable that this generalization is true, but I have not as yet been able to prove it.

In order to show the linear relations between the squares of the P-functions with quadruple indices, we will begin with the 84 groups, of which

$$kl \quad lm \quad mk \quad klm \quad klmn \quad klmp \quad klmq \quad klmr \quad klms \quad klmt$$

is the first. Take the functions

$$P_{klm}, \quad P_{klmn}, \quad P_{klmp}, \quad P_{klmq},$$

and find the values of A, B, C, D, so that the radicals $\sqrt{R(x_1)R(x_2)}$, etc., shall vanish in the sum $AP^2_{klm} + BP^2_{klmn} + CP^2_{klmp} + DP^2_{klmq}$.

Dropping out a common factor

$$[a_k][x_3, x_4][a_l][x_3, x_4][a_m][x_3, x_4]$$

the necessary condition is easily seen to be

$$A + B[a_n][x_3, x_4] + C[a_p][x_3, x_4] + D[a_q][x_3, x_4] = 0;$$

this is equivalent to

$$B + C + D = 0,$$
$$a_n B + a_p C + a_q D = 0,$$
$$A + a_n^2 B + a_p^2 C + a_q^2 D = 0,$$

and these are easily seen to be satisfied by the values

$$A = -|a_n a_p a_q|, \quad B = |a_p a_q|, \quad C = |a_q a_n|, \quad D = |a_n a_p|.$$

Introducing these values in the above sum we have

$$-|a_n a_p a_q| P^2_{klm} + |a_p a_q| P^2_{klmn} + |a_q a_n| P^2_{klmp} + |a_n a_p| P^2_{klmq}$$

$$= \frac{1}{|x_1 x_2 x_3 x_4|^2}$$
$$\{-|a_n a_p a_q|[|x_2 x_3 x_4|^2 [a_n, a_p, a_q, a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4][a_m][x_2, x_3, x_4] + \ldots]$$
$$+ |a_p a_q|[|x_2 x_3 x_4|^2 [a_p, a_q, a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4][a_m][x_2, x_3, x_4][a_n][x_2, x_3, x_4] + \ldots]$$
$$+ |a_q a_n|[|x_2 x_3 x_4|^2 [a_q, a_r, a_s, a_t, a_n][x_1] \text{ `` `` `` `` `` `` `` `` }] + \ldots]$$
$$+ |a_n a_p|[|x_2 x_3 x_4|^2 [a_r, a_s, a_t, a_n, a_p][x_1] \text{ `` `` `` `` `` `` `` `` }] + \ldots]\}$$

The right-hand side of this equation may be written in the form

$$\frac{1}{|x_1 x_2 x_3 x_4|^2} [|a_p a_q||x_2 x_3 x_4|[a_r, a_s, a_t, a_p, a_q][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4][a_m][x_2, x_3, x_4]$$
$$\times \{|x_2 x_3 x_4|[a_n][x_2, x_3, x_4] - |x_3 x_4 x_1|[a_n][x_3, x_4, x_1] + |x_4 x_1 x_2|[a_n][x_2, x_3, x_4]$$
$$- |x_1 x_2 x_3|[a_n][x_1, x_2, x_3]\} + \ldots$$
$$+ \text{ (two similar terms containing } [a_q, a_n][x_1], [a_n, a_p][x_1] \text{ respectively)}$$
$$- |a_n a_p a_q|[a_r, a_s, a_t, a_n, a_p, a_q][x_1][a_k][x_2, x_3, x_4][a_i][x_2, x_3, x_4][a_m][x_2, x_3, x_4]$$
$$\times \{|x_2 x_3 x_4| - |x_3 x_4 x_1| + |x_4 x_1 x_2| - |x_1 x_2 x_3|\} + \ldots]$$

The omitted terms are easily supplied by symmetry. Now

$$\Sigma|x_2 x_3 x_4| = 0, \quad \Sigma|x_2 x_3 x_4|[a_n][x_2, x_3, x_4] = |x_1 x_2 x_3 x_4|$$

The fourth line of the last equation vanishes and the first terms of the first three

$$|x_2 x_3 x_4|[a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4][a_m][x_2, x_3, x_4]$$

for a common factor; similar factors exist of course for the remaining nine terms (taken three at a time) of the first three lines. This factor is to be multiplied by

$$|a_p a_q||a_p x_1||a_q x_1| + |a_q a_n||a_q x_1||a_n x_1| + |a_n a_p||a_n x_1||a_p x_1|, = |a_n a_p a_q|$$

We have then for the right-hand side of 58 the value

$$59.\quad \frac{1}{|x_1x_2x_3x_4|}|a_n a_p a_q|\Sigma|x_2x_3x_4|[a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4][a_m][x_2, x_3, x_4]$$

It is quite easy to show that the quantity under the sign of summation contains θ, *i. e.* $|x_1x_2x_3x_4|$ as a factor. In fact replacing a_k, a_l, a_m by a, b, c respectively and writing

$$\alpha_1 = x_2 + x_3 + x_4,$$
$$\beta_1 = x_2x_3 + x_3x_4 + x_4x_2,$$
$$\gamma_1 = x_2x_3x_4,$$

with similar expressions for α_2, β_2, γ_2, etc., obtained by advancing the suffixes of the x's, we have for the expansion of $[a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4][a_m][x_2, x_3, x_4]$ the following

1	$+ a^3b^3c^3$
α_1	$- \Sigma a^2b^3c^3$
α_1^2	$+ \Sigma a^2b^2c^3$
β_1	$+ \Sigma ab^3c^3$
α_1^3	$- a^2b^2c^2$
γ_1	$- \Sigma b^3c^3$
$\alpha_1\beta_1$	$- \Sigma ab^2c^3$
β_1^2	$+ \Sigma abc^3$
$\alpha_1\gamma_1$	$+ \Sigma a^2b^3$
$\alpha_1^2\beta_1$	$+ \Sigma ab^2c^2$

$+$

$\beta_1\gamma_1$	$- \Sigma ab^3$
$\alpha_1^2\gamma_1$	$- \Sigma a^2b^2$
$\alpha_1\beta_1^2$	$- \Sigma abc^2$
β_1^3	$+ abc$
γ_1^2	$+ \Sigma a^3$
$\alpha_1\beta_1\gamma_1$	$+ \Sigma ab^2$
$\beta_1^2\gamma_1$	$- \Sigma ab$
$\alpha_1\gamma_1^2$	$- \Sigma a^2$
$\beta_1\gamma_1^2$	$+ \Sigma a$
γ_1^3	$- 1$

The second columns here are the coefficients of the quantities in the first columns. Form now the sum

$$\Sigma|x_2x_3x_4|[a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4][a_m][x_2, x_3, x_4].$$

Neglecting the coefficients depending upon a, b, c, we have the identities

$$\begin{aligned}
\Sigma\,|x_2x_3x_4| &= 0 \\
\Sigma\alpha_1\,|x_2x_3x_4| &= 0 \\
\Sigma\beta_1\,|x_2x_3x_4| &= 0 \\
\Sigma\alpha_1^2\,|x_2x_3x_4| &= 0 \\
\Sigma\alpha_1^3\,|x_2x_3x_4| &= -\theta \\
\Sigma\gamma_1\,|x_2x_3x_4| &= -\theta \\
\Sigma\alpha_1\beta_1\,|x_2x_3x_4| &= -\theta \\
\Sigma\beta_1^2\,|x_2x_3x_4| &= -\alpha\theta
\end{aligned}$$

$$\begin{aligned}
\Sigma\alpha_1\gamma_1\,|x_2x_3x_4| &= \alpha\theta \\
\Sigma\alpha_1^2\beta_1\,|x_2x_3x_4| &= -\alpha\theta \\
\Sigma\beta_1\gamma_1\,|x_2x_3x_4| &= -\beta\theta \\
\Sigma\alpha_1^2\gamma_1\,|x_2x_3x_4| &= -\beta\theta \\
\Sigma\alpha_1\beta_1^2\,|x_2x_3x_4| &= \beta\theta
\end{aligned}$$

and so on to

$$\Sigma\gamma_1^3\,|x_2x_3x_4| = -(\gamma^2-2\beta\delta)\,\theta$$

It is obvious then that θ is a factor of the quantity under the summation sign in 59. Beyond this point I have not, as yet, been able to go. On reducing 59, we ought, if the general theorem stated above be true, to find that it is equal to the sum of the squares of two of the functions P_{kl}, P_{lm}, P_{mk}, each multiplied by a constant coefficient depending upon the quantities a. But from the symmetry of the thing we can see *à priori* that there is no reason why any particular pair of these functions should come in more than any other pair. I have made two or three reductions of 59, but have not as yet been able to arrive at any interpretable result.